

Singular Lorentz-Violating Lagrangians and Associated Finsler Structures.

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Several lagrangians associated to classical limits of lorentz-violating fermions in the Standard Model extension (SME) have been shown to yield Finsler functions when the theory is expressed in Euclidean space. When spin-couplings are present, the lagrangian can develop singularities that obstruct the construction of a globally defined Legendre transformation, leading to singular Finsler spaces. A specific sector of the SME where such problems arise is studied. It is found that the singular behavior can be eliminated by an appropriate lifting of the problem to an associated algebraic variety. This provides a smooth classical model for the singular problem. In Euclidean space, the procedure involves combining two related singular Finsler functions into a single smooth function with a semi-positive definite quadratic form defined on a desingularized variety.

I. INTRODUCTION

The potential for breaking of Lorentz symmetry in physics underlying the standard model has been proposed in a variety of contexts, including a promising mechanism for spontaneous violation arising within string field theory [1]. The Standard Model Extension (SME) involves general Lorentz-violating parameters that can arise from such an underlying theory [2]. The theory is formulated in the framework of relativistic quantum field theory which involves a natural expression in momentum space that leads to dispersion relations that can modify particle propagation in the classical limit. The usual procedure for obtaining a classical limit involves performing a Foldy-Wouthuysen transformation on the underlying Hamiltonian and identifying the new coordinate operator in the resulting representation as the relevant classical position operator. At this point, the theory is formulated in terms of momentum, whereas the classical trajectories are measured in terms of the rate of change of the expectation value of the new position operator, referred to as the classical particle velocity. To determine the classical trajectories, it is therefore desirable to find a classical effective lagrangian for the theory. A good analogy is the ray optics limit of Maxwell equations, a formulation of tremendous value and simplification when the wave nature of light is largely irrelevant. Such a project was initiated in [3] with the successful implementation of the an exact Legendre transformation for some special choices of Lorentz-violation parameters in the fermion sector of the SME. Since then, several other papers have presented various other exactly-solvable cases and limits [4]. It was noticed [5] that when converted to Euclidean space, these lagrangians generated a variety of Finsler functions, some of which were singular. It was also pointed out that Finsler space as a generalization to Riemann space may be a way to evade the "no go" theorem of including explicit symmetry breaking into general relativity theories [7].

In simple cases where spin couplings are irrelevant, either modified Minkowski or Randers spaces are found to emerge. On the other hand, spin-dependent couplings produce multiple-valued lagrangians that lead to a set of singular Finsler functions. Singular sets are generically present in these functions where the resulting metrics can diverge and impede the construction of a global Finsler geometry. In the original paper [5], these singular sets were simply removed from the space leaving a singular Finsler space [6] defined on the remaining open subsets. There are several undesirable features of this approach, the most obvious one being that the resulting space is not complete, so the particles described by the corresponding lagrangian are forbidden to travel with certain velocities. In some cases,

these velocities are not actually attainable and are irrelevant physically, but in other cases they can be easily accessible creating a serious impediment to formulating the full theory in terms of Finsler geometry.

A plot of the indicatrix of each singular Finsler function generally reveals cusps at the singular sets. When the singular Finsler functions associated with a specific Lorentz-violating parameter are expressed in terms of a single algebraic variety, the cusps on the indicatrix are replaced by singularities on the variety. In this work, we apply a desingularization procedure to resolve the singular points. The "Finsler b-space" resulting from one of the simplest spin-dependent couplings is analyzed in detail. Note that a recent paper has constructed some interesting classical physics models that lead to these Finsler functions [8] providing intuition about some properties of the space.

II. FINSLER b-SPACE

One of the first spin-dependent terms in the SME to yield a relatively simple lagrangian was the term $b^\mu \bar{\psi} \gamma^5 \gamma_\mu \psi$. The classical lagrangian corresponding to this field-theoretic term is calculated by performing a Legendre Transformation of the dispersion relation [3], with result

$$\mathcal{L}_\pm = -m\sqrt{u^2} \mp \sqrt{(b \cdot u)^2 - b^2 u^2}, \quad (1)$$

where the invariant product was taken to be flat Minkowskian and the b^μ a constant vector field. In a subsequent work [5], the theory was extended to a more general setting where the invariant product is determined by a pseudo-Riemannian metric $r_{\mu\nu}(x)$ and the one-form $b^\mu(x)$ was extended to a general function of x . This procedure is most likely to work if the fields are slowly varying over spacetime so that effects of derivatives of the Lorentz-violating background fields and the metric can be neglected in the Foldy-Wouthuysen transformation that leads to the classical limit. For simplicity, we restrict the presentation here to constant background fields with Minkowski product. The lagrangian can be Wick rotated to Euclidean space yielding the Finsler b-space functions [5]

$$F_\pm = \sqrt{y^2} \pm \sqrt{b^2 y^2 - (b \cdot y)^2}, \quad (2)$$

where y^i are the velocity components in 4 dimensions (which can easily be generalized to n), the mass has been set to unity, and the inner product is Euclidean. It is convenient to separate y into components y_p along b , and y_\perp^i perpendicular to b . The quantity under the second square root sign then reduces to $b^2 y_\perp^2$. If one chooses either F_+ or F_- to compute

the Finsler metrics g^+ and g^- as

$$g_{ij}^{\pm} = \frac{1}{2} \frac{\partial^2 F_{\pm}^2}{\partial y^i \partial y^j}, \quad (3)$$

the resulting metric components in this special coordinate system are

$$g_{00}^{\pm} = 1 \pm \sqrt{\frac{b^2 y_{\perp}^6}{y^6}}, \quad g_{0i}^{\pm} = \pm \sqrt{\frac{b^2 y_p^6}{y^6}} \hat{y}_{\perp}^i, \quad (4)$$

$$g_{ij}^{\pm} = \left(1 + b^2 \pm \sqrt{\frac{b^2}{y^2 y_{\perp}^2}} (y^2 + y_{\perp}^2) \right) \delta^{ij} \mp \sqrt{\frac{b^2}{y^6 y_{\perp}^2}} y_p^4 \hat{y}_{\perp}^i \hat{y}_{\perp}^j \quad (5)$$

where 0 indicates the index along b and $i = \{1, 2, \dots, n-1\}$ are the directions perpendicular to b .

The components in Eq.(5) have singular behavior along the line $y_{\perp} = 0$ as can be seen from the presence of y_{\perp}^2 in the denominators of the g_{ij} -terms. This indicates that neither F_- nor F_+ by itself is sufficient to describe the geometry of a global Finsler space. In fact, the first axiom of Finsler functions requires infinite differentiability away from $y = 0$, a condition that is clearly violated by both F_+ and F_- .

To begin an analysis of the singular behavior of F_{\pm} , recall that in conventional Finsler geometry the Finsler function F scales as $F(\lambda y) = \lambda F(y)$. The scaling property and reparametrization of the distance functional

$$D_F = \int F(y(\lambda)) d\lambda, \quad (6)$$

imply that any path can be reparamterized to lie on the level set defined by $F = 1$. The corresponding hypersurface is called the *indicatrix*: the indicatrix suffices to investigate the geometry of the Finsler space.

To proceed, we plot both F_+ and F_- and construct an associated indicatriix for each, the resulting plot exhibits smooth paths from the hypersurface associated to F_- to the hypersurface associated to F_+ at the points where the geometry of either one becomes singular. This suggests that the F_+ and F_- should be considered as arising from a single algebraic variety that might be desingularized prior to root extraction. It is then consistent to impose the condition $F = 1$ with the result that the variety is now a double-cover of the sphere. This is checked explicitly below.

Squaring twice to eliminate the square roots in Eq.(2) yields the polynomial condition $f(F, y_p, y_{\perp}^i) = 0$ that defines an algebraic variety $X \subset \mathbb{R}^{n+1}$, with

$$f(F, y_p, y_{\perp}^i) = (F^2 - y^2)^2 - b^2 y_{\perp}^2 (2(F^2 + y^2) - b^2 y_{\perp}^2) = 0. \quad (7)$$

The variety is invariant under the transformation $(F \rightarrow \lambda F, y^i \rightarrow \lambda y^i)$, the generalization of the Finsler function homogeneity condition. Note that the gradient of f vanishes when $y_\perp^i = 0$, indicating the presence of a singular set Σ , a line in the variety X . In particular, X is not a smooth manifold as it stands. Note that the above procedure introduces $F < 0$ solutions, however, these do not intersect the $F > 0$ variety and can therefore be treated independently. The indicatrix can now be constructed by setting $F = 1$, now possible since F is expressed in terms of the variety X . This results in the constraint

$$y^2 = \left(1 \mp \sqrt{b^2 y_\perp^2}\right)^2, \quad (8)$$

on the y^i variables, and the two solutions correspond to the two roots F_\pm . Eq.(8) explicitly exhibits the indicatrix as a double-valued, small perturbation from the half-circle in the $(|y_\perp|, y_p)$ half-plane. The singular points result at the poles $y_\perp = 0$ where the derivatives $y_p'(|y_\perp|) \rightarrow \pm 2|b|$ fail to vanish, a condition required for the hypersurface of revolution (symmetric in the y_\perp^i) to be smooth.

III. FORMAL DESINGULARIZATION

For simplicity in notation, only the singular points where $F > 0$ and $y_p > 0$ are considered in what follows, as the others can be handled similarly using the symmetries of the defining variety. To desingularize the variety, define a new coordinate u^i so that

$$y_\perp^i = (F^2 - y_p^2)u^i \quad (9)$$

and study the new variety in a small neighborhood of the singular point. Plugging Eq.(9) into the variety equation yields $f = (F^2 - y_p^2)^2 h(F, y_p, u^i)$, where

$$h = (1 - (F^2 - y_p^2)u^2)^2 - b^2 u^2 [2(F^2 + y_p^2) + (2 - b^2)(F^2 - y_p^2)^2 u^2], \quad (10)$$

and $h(F, y_p, u^i) = 0$ yields the same variety away from the singular set. The exceptional locus $F^2 = y_p^2$ intersects the variety $h = 0$ on a sphere (for fixed y_p) with

$$u^2 = \frac{1}{4b^2 y_p^2}, \quad (11)$$

demonstrating that the new u^i variables are not all identically zero at the singular points. A new desingularized variety \tilde{X} can then be defined by first removing all points in a small neighborhood of the singular set from X and then "gluing in" a copy of $S^{n-2} \times \mathbb{R}$, where the

line \mathbb{R} is just y_p and S^{n-2} is the sphere determined by u^i at the appropriate fixed y_p value. Then \tilde{X} admits a smooth, differentiable structure.

To identify the neighborhood about the singular point for which the u^i coordinates are valid, general solutions to $F^2 - y_p^2 = 0$ can be examined. In addition to $y_\perp^i = 0$, there is another solution on the F_- sheet given by

$$y_\perp^2 = \frac{4b^2}{(1-b^2)^2} y_p^2 \equiv \epsilon^2. \quad (12)$$

The neighborhood in which the u^i coordinates give a smooth structure is therefore restricted to the region $y_\perp^2 < \epsilon^2$ so that the transformation (9) is non-singular. This also happens to be precisely the region where F_- fails to be convex. Note that the gradient of h is nonzero everywhere in this region with the singular set lifting to a sphere in the new coordinates. This is commensurate with what is expected from the spin variable in $n = 4$ dimensions where the singular set corresponds to a two-sphere on which the classical spin of the particle points.

The variables u^i can be thought of as auxiliary variables that specify a smooth gluing of the two sheets present after the singular set is removed. Expressing u^i in terms of y^i gives

$$u^i = \frac{y_\perp^i}{(F^2 - y_p^2)} = \frac{y_\perp^i}{(1+b^2)y_\perp^2 \pm 2|b||y_\perp|\sqrt{y^2}}, \quad (13)$$

yielding two solutions, one for the outer sheet determined by F_+ and the other for the inner sheet determined by F_- . At the singular point u^i is the vector that points along y_\perp^i for the positive choice and opposite to y_\perp^i for the negative choice. A continuous curve $\gamma(t)$ in the variety \tilde{X} must therefore change sheets in X as it passes through a singular point so that $u^i(t)$ remains continuous as y_\perp^i necessarily changes sign.

IV. IMPROVED COORDINATES AND A NEW METRIC

Near the singular point, the metric calculated using the y^i variables diverges as is seen in Eq. (5). It is natural to ask if the u^i variables can be used to define a finite, consistent metric in the neighborhood of the singular point. Solving for F in terms of the u -variables gives

$$\tilde{F}_\pm(y_p, u^i) = \frac{1}{(1-b^2)\sqrt{u^2}} \left[\sqrt{1 + (1-b^2)^2 u^2 y_p^2} \pm |b| \right]. \quad (14)$$

Only the lower sign has the correct limit at the singular point, so it can be deduced that $F_u = \tilde{F}_-$ is the relevant Finsler function. Note that the Finsler function as expressed in terms

of the new variables is in fact smooth and single-valued near the singular point as expected from the desingularization. Unfortunately, $F_u(y_p, u^i)$ is not a homogeneous function of its new variable set as $u^i \rightarrow (1/\lambda)u^i$ when $y^i \rightarrow \lambda y^i$, so it is not possible to use the conventional argument to define a Finsler metric. Fortunately, this problem can be remedied by defining a new variable

$$w^i = \frac{u^i}{u^2} = \frac{F^2 - y_p^2}{y_\perp^2} y_\perp^i, \quad (15)$$

with the same scaling properties as y_i . Then F_u becomes a homogeneous function in terms of w

$$F_u = \sqrt{y_p^2 + \frac{w^2}{(1-b^2)^2}} - \sqrt{b^2 \frac{w^2}{(1-b^2)^2}}. \quad (16)$$

One further obvious scaling $w^i = (1-b^2)z^i$ brings the Finsler function into the same form as F_- given by Eq.(2), one of the original functions we started with. In fact, the transformation of variables is a symmetry of the original polynomial condition $f(F, y_p, y_\perp^i) = 0$ as can be easily verified by direct substitution of $u(z)$ into $h(F, y_p, u^i)$

$$h(F, y_p, u^i(z)) = \frac{1}{(1-b^2)^2 z^4} [(F^2 - y_p^2 - z^2)^2 - b^2 z^2 (2(F^2 + y_p^2 + z^2) - b^2 z^2)], \quad (17)$$

where the factor in brackets is just $f(F, y_p, z^i)$, the function defined in Eq.(7) with $y_\perp^i \rightarrow z^i$.

A relevant fact is that the original singular point $y_\perp^i = 0$ maps to a sphere of radius $\epsilon = 2|b|y_p/(1-b^2)$ in the z^i variables, while the second solution to $F_-^2 = y_p^2$ given in Eq.(12) for y_\perp gets mapped to a singular set in the new coordinates, Σ' , defined by $z^i = 0$. The metric can now be computed exactly as it is for F_- in terms of z and gives a finite, well-defined value everywhere on this sphere. Writing z^i in terms of the original variables using F_\pm yields

$$z^i = \frac{1}{1-b^2} \left(\frac{F^2 - y_p^2}{y_\perp^2} \right) y_\perp^i = \frac{1}{1-b^2} \left[1 + b^2 \pm 2\sqrt{\frac{b^2 y_p^2}{y_\perp^2}} \right] y_\perp^i, \quad (18)$$

demonstrating explicitly that z^i survives at the singular point $y_\perp^i \rightarrow 0$ with

$$z^2 \rightarrow \frac{4b^2 y_p^2}{(1-b^2)^2}, \quad (19)$$

and direction parallel to the unit vector along y_\perp^i as the limit is taken. Calculation of the metric using $F_-(y_p, z^i)$ on the singular set yields

$$\tilde{g}_{pp} = 1 - \frac{8b^4}{(1+b^2)^3}, \quad \tilde{g}_{pi} = -|b| \frac{(1-b^2)^3}{(1+b^2)^3} \hat{z}^i, \quad \tilde{g}_{ij} = \frac{(1-b^2)^2}{2(1+b^2)} \left[\delta_{ij} + \frac{(1-b^2)^2}{(1+b^2)^2} \hat{z}^i \hat{z}^j \right]. \quad (20)$$

Note that the result is finite and well-defined and depends only on the unit vector \hat{z} on the sphere.

Together, the two charts on \tilde{X} defined using $\{F, y_p, y_\perp^i\}$ on $X - \Sigma$ and $\{F, y_p, z^i\}$ on $X - \Sigma'$ form an atlas for the de-singularized variety \tilde{X} . Note that it is always possible to choose one or the other set of coordinates to enforce the condition $\det g \geq 0$. The only issue is that there exists a sphere in the space (on F_- at $y_\perp = \sqrt{1-b^2} \epsilon/2$ in terms of the y^i variables) where $\det g$ is identically zero for either choice of charts. This indicates that the resulting global Finsler structure is only positive semi-definite, not strictly positive definite. The physical meaning of this result is discussed in the next section.

V. LAGRANGIAN

The Euclidean structure can be converted back to the original Minkowski structures from which they were derived by performing a Wick rotation where $n = 4$ dimensions are used and the time-components of the four-vectors are multiplied by i . The squares and dot-products of the four-vectors convert over to their Minkowski counterparts with a sign. For example, $y^2 \rightarrow -u^2$ where $u^2 = (u_0)^2 - \vec{u}^2$ depends on the Minkowski metric. Under this map, $F \rightarrow -iL$, and $y_p \rightarrow u_p$, $y_\perp \rightarrow u_\perp$ where the parallel and perpendicular components of u are determined by the Minkowski metric. Explicitly, $u^\mu = u_p^\mu + u_\perp^\mu$ with (note that this map only works when $b^2 \neq 0$)

$$u_\perp^\mu = u^\mu - \frac{u \cdot b}{b^2} b^\mu. \quad (21)$$

The lagrangian per unit mass (with b^μ in units of the mass m) becomes

$$L_\pm = -\sqrt{u^2} \mp \sqrt{b^2(-u_\perp^2)}. \quad (22)$$

The desingularization variable z^i maps to

$$z^\mu = \frac{1}{1+b^2} \left[\frac{L^2 - u_p^2}{u_\perp^2} \right] u_\perp^\mu = \frac{1}{1+b^2} \left[1 - b^2 \pm 2\sqrt{\frac{b^2 u^2}{-u_\perp^2}} \right] u_\perp^\mu. \quad (23)$$

The condition that there is a singular point on the variety is now that $u_\perp^\mu = 0$. When b^μ is spacelike ($b^2 < 0$), the singular point lies in a space-like velocity region that is inaccessible to physical particles making the singular points largely irrelevant.

When b^μ is time-like ($b^2 > 0$), a Lorentz transformation can be used to go to a frame in which only $b^0 \neq 0$. In this frame, the singular point is where the three-velocity of the particle vanishes, $\vec{u} = \vec{v} = 0$. Attention is therefore focused on the special case $b^\mu = (b^0, 0, 0, 0)$ so that the singular point lies at $\vec{v} = 0$ in what follows. In addition, standard proper time parametrization is assumed so that $u^2 = 1$ can be imposed. In this case, the sign appearing

in the definition of L_{\pm} indicates the velocity-helicity of the particle, and it must be prescribed in order to define a particle's trajectory. Note that velocity-helicity and momentum-helicity can be different due to the modification of the momentum-velocity relation

$$p^i = \gamma v^i \mp b_0 \hat{v} \quad (24)$$

It is not surprising that the metric breaks down at $\vec{v} = 0$ since the three-momentum tends to $\vec{p} \rightarrow \mp b_0 \hat{v} \neq 0$, so the direction of the momentum is not determined there. The desingularization variable at the singular point is

$$z^{\mu} \rightarrow \left(0, \pm \frac{2b_0}{1+b_0^2} \hat{v} \right), \quad (25)$$

where \hat{v} is a unit vector that can point in any spatial direction. The positive and negative values for z can be associated with the velocity-helicity of the particle, therefore the sign is determined by the particle's spin direction, $\pm \hat{v}$. If \vec{z} is required to vary continuously as a particle moves through the singular point (by stopping and reversing direction) then the momentum must also remain continuous along the trajectory indicating that the transition from L_{\pm} to L_{\mp} is required.

The region where the Finsler metric of the Euclidean case fails to be positive definite corresponds to the low-speed region where $\vec{v}^2 \leq b_0^2/(1+b_0^2)$ and the momentum is in fact opposite the velocity. In this region it is possible to increase the particle's momentum while decreasing its velocity and energy ($p^0 = \gamma m$ can be expressed purely in terms of the velocity).

Another curious implication of the vanishing of certain eigenvalues of the metric in directions orthogonal to the particle velocity is the existence of extremal action solutions with $\vec{p} = 0$ that have nonzero velocity. Arbitrary changes in the direction of the velocity do not change the action indicating that the direction of \vec{v} can randomly vary as the particle moves along. The particle can even exhibit uniform circular motion with zero force. These types of motion should be considered spurious and indicate a problem with the model, presumably due to the fact that spin has been fixed to helicity eigenstates in this limit of the full quantum theory. Desingularization can smooth out the variety at the singular point and indicate the source of these spurious trajectories. At the singular point it is not sufficient to specify the velocity but necessary to also retain some unit vector direction proportional to the momentum vector. This suggests there is an additional internal variable in the system determining the direction of this momentum, namely the particle spin. Requiring that the particle spin angular momentum be conserved is one way to eliminate these spurious

trajectories.

VI. GENERAL PROCEDURE

In this section, we make several remarks concerning a general procedure that could be used to perform the desingularization within the context of the general derivation presented in [3]. The procedure outlined in [3] starts with a general SME fermion dispersion relation that yields a generic polynomial condition $P(L) = 0$, where the coefficients in the polynomial depend on u^μ , m , and the Lorentz-violating parameters. The solutions of this equation determine the possible lagrangians as perturbations of the conventional $L = \pm m\sqrt{u^2}$ structure. The four degrees of freedom of the relativistic theory split into two particle and two anti-particle states according to the overall sign of this Lagrangian. This fact can be deduced by examination of the defining equation $L = -p_\mu u^\mu$ in the rest frame where $\vec{u} = 0$, recalling that the negative-energy solutions are reinterpreted at the level of the quantum field theory as antiparticles. The desingularization procedure is applied to the corresponding variety $f(F, y^i) = 0$ obtained from $P(L) = 0$ by using an appropriate Euclideanization procedure, like Wick rotation. The singular points are then determined by finding points where the gradient of f is zero. An appropriate blowup procedure should then be used to produce a smooth manifold that sits above the variety. In this paper, the blowup is performed using an additional set of coordinates near the singular point that are related to the slopes of lines through the singular point. Using appropriate new coordinates in the region where the original metric fails to be positive definite, it is possible to define a new metric that is positive semi-definite everywhere on the manifold. In general, finding an appropriate set of "new coordinates" near the singular points should be possible. The existence of suitable coordinates that make the resulting quadratic form semi-positive definite is an interesting open issue.

In addition to the physical solutions, there are often additional, spurious solutions to $P(L) = 0$ that yield nonperturbative values for the lagrangian that do not have the correct standard limit as the Lorentz-violation parameters are tuned to zero. For example, in the b^μ -case, $P(L)$ is an octic polynomial and there is an additional solution $L = -\alpha\sqrt{(b \cdot u)^2}$, with $\alpha = \sqrt{m^2/b^2 + 1}$ that leads to a zero metric in the corresponding Finsler space. The indicatrix is the flat hypersurface $y_p = 1/\sqrt{1 - b^2}$ that is tangent to the top of the $F_- = 1$ singular Finsler function. In fact, this solution can be applied in the region where F_- fails to be convex, rather like the Gibbs construction of thermodynamics. Physically, the spurious

lagrangian corresponds to particles that have zero velocity and the fixed momentum value $p_\mu = \alpha \text{sign}(b \cdot u)b_\mu$. Applying this "Gibbs construction" to time-like $b \rightarrow (b^0, 0, 0, 0)$ case in the region where F_- fails to be convex yields a region at low velocities where the particle with negative velocity-helicity gets mapped to a zero velocity state. In this case, the badly-behaved region is simply collapsed to a point. Despite this curious mathematical possibility, it seems unlikely to correspond to a realistic model as the spin would somehow have to constantly adjust to keep the velocity zero at these very low velocities violating spin angular momentum conservation.

VI. CONCLUSIONS

Desingularization of the lagrangian variety that arises from the SME b^μ -coupling can be accomplished by first converting to Euclidean space where the theory is determined by a pair of singular Finsler functions. These functions are defined using an algebraic variety which is then desingularized using a new parametrization of the variety. This procedure yields an implicitly determined F on an underlying manifold. The fact that F is determined implicitly by a variety condition and is generally double-valued violates the first axiom of Finsler geometry which states that F must be a function. In addition, there are regions where the metric becomes negative definite near the singular point. It turns out to be possible to parameterize the "badly behaved" It would be natural to generalize the concept of a Finsler geometry to allow an appropriate formulation on the lifted variety (a manifold), but for the single remaining impediment due to the degeneracy of the metric at very specific values of the velocity. This degeneracy corresponds to a spurious set of extremal paths at low velocity. In the physical case, these paths might be inaccessible to the particle (as is the case for space-like b) or may be eliminated by requiring conservation of spin, a variable that has been discarded in the transition to the classical model. The combination of the two charts forms an atlas on the resulting manifold that allows identification of smooth paths through the singular set in the original variety. The result is a smooth classical model on a manifold that sits above the variety that can be used for practical calculations.

The explicit construction for the b^μ case presented in this paper suggests that it is possible to desingularize classical lagrangians defined on singular algebraic varieties through standard procedures yielding consistent models for the classical ray limit of Lorentz-violating theories. Such a model would yield a consistent formulation of classical particle propagation within the context of general relativity when explicit Lorentz violation is present in the matter sector.

Some generalization of pseudo-Riemann-Finsler geometry which involves a formulation on a variety and allows for degenerate metric directions inside the light cone appears to be required to consistently describe these models in a geometric way. These new requirements are a fundamental implication of including spin coupling into the theory as a physically relevant quantity.

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